Subgroups

Properties of a group can be studied in terms of its internal structure. Some groups contain subgroups of interest. A subgroup is a group within a group; just as a subset is a set within a set.

Subgroup defined:

> A non-null set H is said to be a subgroup of a group G iff:
>
> (1) every element of H is an element of G
>
> (2) H is a group under * (the binary operation of G)

The term proper subgroup is analogous to the term proper subset.

Some additional language before displaying subgroups of a perm. group.

1) The group of all mappings of a set onto itself is called a symmetric group.

2) The number of elements in the set from which the group is formed is called the <u>degree</u> of the group.

3) The number of elements in the group itself is called the order of the group.

4) Thus, for example, the permutation group formed from the set 0,1,2 is of degree 3 and order 6.

Interpretation of order-inversions with respect to the structure of a subgroup

[subgroup of even permutations (2)(2) in the group of degree four] -- next page

Chap. 4: Order Relations

Technical background -- topics

- Permutations
  - Number of ordered and unordered sets
  - Permutation as mapping
  - Measuring rearrangement & (order inversions)
  - Permutation multiplication
  - One-one notation and cyclic notation
  - permutation group
    - subgroup
    - parity
    - partitions and permutations
    - inverse-related permutations
- Group theory - further aspects (see group theory folder)
  - cosets
  - conjugate classes

As an example, consider the pc set $S = 6, 11, 5, 0$ (a compositional set)

The order of elements for this set can be indicated by subscripts, thus:

$$6_1 \; 11_2 \; 5_3 \; 0_4$$

Suppose there is another occurrence of this set, ~~xxtxxx~~

$S_1 = 11, 0, 5, 6$

We see immediately that this is the same as the first occurrence with respect to pitch content. We can examine the amount of rearrangement of S_1 with respect to S by attaching the original subscripts, thus:

$$11_2 \; 0_4 \; 5_3 \; 6_1$$

by comparing the subscripts

```
1 2 3 4
2 4 3 1
```

we see that the first element has been replaced by the second, the second by the fourth, the third by the third and the fourth by the first.

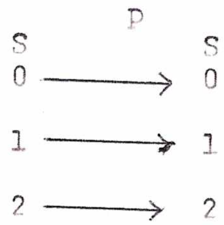
This pattern of rearrangement is interesting. In order to "understand" it more completely it would be of interest to know or to be able to measure in some way the amount of derangement it represents, how it compares with other possible rearrangements (permutations) and so on. To answer these questions we require some tools.

Permutation defined:

~~A permutation~~

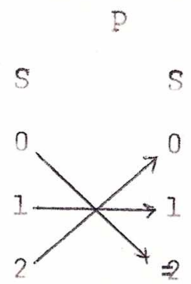
A permutation p of a set S is a one-to-one mapping of S onto itself. The range of p is the same as the domain of p.

Example:



Here p maps each element of S onto itself.

Another example:



Here p maps 0 onto 2, 1 onto 1 and 2 onto 0.

For a set of n elements there are n! permutations.

In thinking about permutations it is important to bear in mind that the "objects" permuted are of secondary significance. The way in which they are rearranged (i.e., the process of reordering) is of primary significance. [continued on next page]

4-23

	Pattern	Permutations				
1)	2-3-2	0 2 5 7	7 5 2 0	P_1 (0)(1)(2)(3)	P_{24} (0 3)(1 2)	EVEN
2)	5-2-3	7 0 2 5	5 2 0 7	P_2 (0 1 2 3)	P_{13} (0 2)	ODD
	3-2-5	2 5 7 0	0 7 5 2	P_3 (0 3 2 1)	P_{12} (1 3)	ODD
3)	2-5-2	5 7 0 2	2 0 7 5	P_{23} (0 2)(1 3)	P_{22} (0 1)(2 3)	EVEN
		0 2 7 5	5 7 2 0	P_{10} (2 3)	P_6 (0 3 1 2)	ODD
		2 0 5 7	7 5 0 2	P_8 (0 1)	P_4 (0 2 1 3)	ODD
4)	5-2-5	5 0 2 7	7 2 0 5	P_{17} (0 1 2)	P_{15} (0 2 3)	EVEN
		0 5 7 2	2 7 5 0	P_{18} (1 3 2)	P_{16} (0 3 1)	EVEN
5)	5-5-5	2 7 0 5	5 0 7 2	P_7 (0 2 3 1)	P_5 (0 1 3 2)	ODD
6)	5-3-5	7 2 5 0	0 5 2 7	P_{11} (0 3)	P_9 (1 2)	ODD
		2 5 0 7	7 0 5 2	P_{21} (0 2 1)	P_{20} (0 1 3)	EVEN
		0 7 2 5	5 2 7 0	P_{14} (1 2 3)	P_{19} (0 3 2)	EVEN

4-9 0,1,6,7

EXAMPLE OF EQUIV. INTERVAL SUCCESSIONS

INTERVAL PATTERN		PERMUTATION		
115	1-5-1	0167 —R—	7610	
		6701 —	1076	(02)(13)
155	5-1-5	7016	6107	
		1670	0761	
116	1-6-1	0176	6710	
		7601	1067	
566	6-5-6	1706	6071	
		7160	0617	
166	6-1-6	6017	7106	
		0671	1760	
556	5-6-5	1607	7061	
		0716	6170	

1-5-1
1-6-1
5-1-5
5-6-5
6-1-6
6-5-6

1-5-1
5-1-5
1-6-1
6-1-6
5-6-5
6-5-6

Patterns of interval succession for pc set 4-19

Vector: [101310]

	Permutation		PC Set	Interval Succession	
p 1	(0)(1)(2)(3)	even	0 1 4 8	1 3 4 ~~4~~	134
2	(0123)		1 4 8 0	3 4 4 ~~1~~	344
3	(0321)		8 0 1 4	4 1 3 ~~4~~	134
4	(0213)		4 8 1 0	4 5 1 ~~4~~	145
5	(0132)	odd	1 8 0 4	5 4 4 ~~3~~	445
6	(0312)		8 4 0 1	4 4 1 ~~5~~	144
7	(0231)		4 0 8 1	4 4 5 ~~3~~	445
8	(01)		1 0 4 8	1 4 4 ~~5~~	
9	(12)		0 4 1 8	4 3 5 ~~4~~	
10	(23)		0 1 8 4	1 5 4 ~~4~~	
11	(03)	odd	8 1 4 0	5 3 4 ~~4~~	
12	(13)		0 8 4 1	4 4 3 ~~1~~	
13	(02)		4 1 0 8	3 1 4 ~~4~~	
14	(123)		0 4 8 1	4 4 5 ~~1~~	
15	(023)		4 1 8 0	3 5 4 ~~4~~	
16	(031)		8 0 4 1	4 4 3 ~~5~~	
17	(012)	even	1 4 0 8	3 4 4 ~~5~~	
18	(132)		0 8 1 4	4 5 3 ~~4~~	
19	(032)		8 1 0 4	5 1 ~~3~~4 ~~4~~	
20	(013)		1 8 4 0	5 4 4 ~~1~~	
21	(021)		4 0 1 8	4 1 5 ~~4~~	
22	(01)(23)		1 0 8 4	1 4 4 ~~3~~	
23	(02)(13)	even	4 8 0 1	4 4 1 ~~3~~	
24	(03)(12)		8 4 1 0	4 3 1 ~~4~~	

Analysis:

6 ~~8~~ basic patterns of interval succession

Pattern	Number of permutations
1 3 4	4
3 4 4	4
1 4 5	~~3~~4
4 4 5	4
1 4 4	4
3 4 5	4
~~1 3 5~~	~~1~~

[How interpret in terms of permutation classes? — subgroups, etc.]

Relation between patterns and vector — see list for 4-25 [020202]

4-6: 0,1,2,7

	Interval-Succession		Permutation				
115	1-1-5	5-1-1	0-1-2-7	7-2-1-0	P_{1} (0) (1)(2)(3)	P_{24} (03)(12)	EVEN 1111 22
	7-1-1	1-1-7	7-0-1-2	2-1-0-7	P_{2} (0123)	P_{13} (02)	ODD 4 211
155	5-7-1	1-7-5	2-7-0-1	1-0-7-2	P_{23} (02) (13)	P_{22} (01) (23)	EVEN 22 22
	1-5-7	7-5-1	1-2-7-0	0-7-2-1	P_{3} (0321)	P_{12} (13)	ODD 4 211
126	2-1-6	6-1-2	2-0-1-7	7-1-0-2	P_{17} (012)	P_{15} (023)	EVEN 3 1
	2-1-6	6-1-2	0-2-1-7	7-1-2-0	P_{9} (12)	P_{11} (03)	ODD 211
256	6-7-2	2-7-6	1-7-0-2	2-0-7-1	P_{7} (0231)	P_{5} (0132)	ODD 4
	6-5-2	2-5-6	1-7-2-0	0-2-7-1	P_{16} (031)	P_{18} (132)	EVEN 3 1
125	1-2-7	7-2-1	1-2-0-7	7-0-2-1	P_{21} (021)	P_{20} (013)	EVEN 3 1
	1-2-5	5-2-1	1-0-2-7	7-2-0-1	P_{8} (01)	P_{4} (0213)	ODD 211 4
156	1-6-5	5-6-1	0-1-7-2	2-7-1-0	P_{10} (23)	P_{6} (0312)	ODD 211 4
	1-6-7	7-6-1	2-1-7-0	0-7-1-2	P_{19} (032)	P_{14} (123)	EVEN 31 31

4-24: 0,2,4,8 [020301]

	Permutation	Permuted Set	Interval-Succession	Interval-Pattern
P_1	(0)(1)(2)(3)	0 2 4 8	2-2-4	224
P_{24}	(03)(12)	8 4 2 0	4-2-2	
P_2	(0 1 2 3)	8 0 2 4	8-2-2	
P_{13}	(02)	4 2 0 8	2-2-8	
P_3	(0 3 2 1)	2 4 8 0	2-4-8	248
P_{12}	(13)	0 8 4 2	8-4-2	
P_4	(0 2 1 3)	8 4 0 2	4-4-2	
P_8	(01)	2 0 4 8	2-4-4	
P_{20}	(0 1 3)	8 0 4 2	8-4-2	
P_{21}	(021)	2 4 0 8	2-4-8	
P_{22}	(01)(23)	2 0 8 4	2-8-4	
P_{23}	(02)(13)	4 8 0 2	4-8-2	
P_5	(0132)	4 0 8 2	4-8-6	468
P_7	(0231)	2 8 0 4	6-8-4	
P_{16}	(031)	2 8 4 0	6-4-4	
P_{18}	(132)	0 4 8 2	4-4-6	
P_6	(0312)	4 8 2 0	4-6-2	246
P_{10}	(23)	0 2 8 4	2-6-4	
P_9	(12)	0 4 2 8	4-2-6	
P_{11}	(03)	8 2 4 0	6-2-4	
P_{14}	(123)	0 8 2 4	8-6-2	
P_{19}	(032)	4 2 8 0	2-6-8	
P_{15}	(023)	8 2 0 4	6-2-4	
P_{17}	(012)	4 0 2 8	4-2-6	

No. of Patterns		No. of pitchsets			Partition
3 = 12-9	4-28	3	0, 3, 6, 9		3 3 3 (3)
4 = 12-8	4-24	12	0, 2, 4, 8		2 2 4 (4)
6 = 12-6	4-1	12	0, 1, 2, 3	[321000]	1 1 1 (9)
	4-6	12	0, 1, 2, 7	[210021]	1 1 5 (5)
	4-9	6	0, 1, 6, 7	[200022]	1 5 1 (5)
	4-19		0, 1, 4, 8	[101310]	1 3 4 (4)
	4-21		0, 2, 4, 6	[030201]	2 2 2 (6)
	4-23		0, 2, 5, 7	[021030]	2 3 2 (5)
	4-25		0, 2, 6, 8	[020202]	2 4 2 (4)
8 = 12-4	4-2		0, 1, 2, 4		1 1 2 (8)
And(R1, Rp)	4-3		0, 1, 3, 4		1 2 1 (8)
Transitive tuples: 4-7, 4-17, 4-20	4-7		0, 1, 4, 5		1 3 1 (7)
	4-8		0, 1, 5, 6		1 4 1 (6)
	4-10		0, 2, 3, 5		2 1 2 (7)
	4-17		0, 3, 4, 7		3 1 3 (5)
	4-20		0, 1, 5, 8		1 4 3 (4)
	4-22		0, 2, 4, 7		2 2 3 (5)
	4-26		0, 3, 5, 8		3 2 3 (4)
10 = 12-2	4-4		0, 1, 2, 5		1 1 3 (7)
And(R1, Rp)	4-5		0, 1, 2, 6		1 1 4 (6)
Transitive tuples: 4-4, 4-11, 4-14	4-11		0, 1, 3, 5		1 2 2 (7)
4-12, 4-13, 4-18, 4-27	4-12		0, 2, 3, 6		2 1 3 (6)
[illegible]	4-13		0, 1, 3, 6		1 2 3 (6)
	4-14		0, 2, 3, 7		2 1 4 (5)
	4-16		0, 1, 5, 7		1 4 2 (5)
	4-18		0, 1, 4, 7		1 3 3 (5)
	4-27		0, 2, 5, 8		2 3 3 (4)
12 = 12-0	4-Z15		0, 1, 4, 6		1 3 2 (6)
	4-Z29		0, 1, 3, 7		1 2 4 (5)

THE FOUR PROPER SUBGROUPS OF THE SYMMETRIC GROUP OF ORDER 6
(THE ONLY SUBGROUPS)

The subgroup of even permutations of a set of DEGREE ~~order~~ 3: THE "ALTERNATING GROUP"
(a normal subgroup - see Papy, p. 101)

$\circ$	π_1	π_2	π_3
π_1	π_1	π_2	π_3
π_2	π_2	π_3	π_1
π_3	π_3	π_1	π_2

This is also a cyclic subgroup, since it can be generated by powers of π_2 or π_3 (an "inverse" pair)

Since each odd permutation is its own inverse, the subgroups generated contain only two elements (below)

Each odd permutation forms a (trivial) subgroup with the identity permutation:
(These are not normal subgroups)

$\circ$	π_1	π_4
π_1	π_1	π_4
π_4	π_4	π_1

$\circ$	π_1	π_5
π_1	π_1	π_5
π_5	π_5	π_1

$\circ$	π_1	π_6
π_1	π_1	π_6
π_6	π_6	π_1

Permutations of degree 5 — pairs by retrogrades

1-1	Cyclic
01234	(0)(1)(2)(3)(4)
43210	(04)(13)
12340	(04321)
04321	(14)(23)
23401	(03142)
10432	(01)(24)
34012	(02413)
21043	(02)(34)
40123	(01234)
32104	(03)(12)
12304	(0321)
40321	(014)(23)
23041	~~(02)~~ (143)(02)
14032	(0241)
30412	(013)(24)
21403	(0342)
04123	(1234)
32140	(043)(12)
41230	(04)
03214	(13)
23014	(02)(13)
41032	(024)
30142	(01243)
24103	(03412)
01423	(234)
32410	(04213)
14230	(041)
03241	(143)
42301	(03214)
10324	(01)(23)
30124	(0123)
42103	(034)(12)
01243	(34)
34210	(0413)
12430	(0421)
03421	(1423)
24301	(032)(14)
10342	(243)(01)

43012	(024)(13)
21034	(02)
12034	(021)
43021	(02314)
20341	(01432)
14302	(03241)
03412	(13)(24)
21430	(042)
34120	(04123)
02143	(12)(34)
41203	(034)
30214	(013)
20314	(0132)
41302	(0324)
03142	(1243)
24130	(0412)
31420	(0423)
02413	(1342)
14203	(0341)
30241	(0143)
42031	(0214)
13024	(0231)
03124	(123)
42130	(04)(12)
31240	(043)
04213	(134)
12403	(03421)
30421	(01423)
24031	(02)(14)
13042	(02431)
40312	(01324)
21304	(032)
31204	(03)
40213	(0134)
12043	(021)(34)
34021	(023)(14)

20431 (0142)
13402 (031)(24)

04312 (1324)
21340 (0432)

43120 (04)(123)
02134 (12)

20134 (012)
43102 (03124)

01342 (243)
24310 (04132)

13420 (04231)
02431 (142)

34201 (03)(14)
10243 (01)(34)

42013 (02134)
31024 (023)(

01324 (23)
42310 (04)(132)

13240 (0431)
04231 (14)

32401 (03)(142)
10423 (01)(234)

24013 (02)(134)
31042 (0243)

40132 (0124)
23104 (0312)

13204 (031)
40231 (014)

32041 (02143)
14023 (02341)

20413 (01342)
31402 (03)(24)

04132 (124)
23140 (04312)

41320 (04)(23)
02314 (132)

32014 (0213)
41023 (0234)

20143 (012)(34)
34102 (03)(124)

01432 (24)
23410 (042)(13)

14320 (041)(23)
02341 (1432)

43201 (0314)
10234 (01)

Cyclic notation for permutations

Permutations are conventionally represented as cycles. This notation has a number of advantages. In particular, it provides the first step toward the reduction of the large number of permutations.

Cyclic notation works as follows:

Consider a ~~set~~ permutation (as "basic" or "prime") 0 1 2

The identity permutation, which maps each element of the set onto itself, is represented as

p_1	0 1 2	(0) (1) (2)

Other permutations are represented in similar fashion

	Permutation	Cyclic notation
p_2	1 2 0	(0 1 2)
p_3	2 0 1	(0 2 1)
p_4	1 0 2 ~~0 2 1~~	~~(1 2)~~ (0 1)
p_5	0 2 1 ~~2 1 0~~	~~(0 2)~~ (1 2)
p_6	2 1 0 ~~1 0 2~~	~~(0 1)~~ (0 2)

Read vertically with reference to identity permutation

The cyclic notation of p_2 is read:"0 is sent onto 1, 1 is sent onto 2, 2 is sent onto 0. Or, more accurately, the first element ~~becomes the~~ is replaced by the second, etc.

Note that for p_4, p_5, and p_6, the single-cycle identity mapping is implied. That is, for p_4 the complete notation would be

p_4 0 2 1 (1 2) (0)

Relations between permutations

(1) Order inversion

The amount of derangement effected by a permutation can be expressed in terms of order inversions.

Arrivers?

Let $d(p_i, p_j)$ represent the number of order reversals of pairs of elements in p_j with respect to the same pairs in p_i.

Example:

$p_1 = 0\ 1\ 2$

$d(p_1, p_1) = 0$

Example:

$p_1 = 0\ 1\ 2$

$p_5 = 2\ 1\ 0$

$d(p_1, p_5) = 3$

Thus, 0 1 in P_1 becomes 1 0 in P5, 0 2 in P_1 becomes 2 0 in p_5, and 1 2 in p_1 becomes 2 1 in p_5.

The last example shows maximum derangement -- commonly known as retrograde.

Interpretation of order-inversions with respect to group structure.

Parity

Every integer has the property "even" (0 or divisible by 2 without remainder) or odd. The oddness or evenness property is referred to as the <u>parity</u> of the number.

Parity applied to permutations:

Any permutation can be decomposed into disjoint cycles and represented in cyclic notation.

already covered Permutation multiplicatbn with cyclic notation [sep. page]

A cycle of 2 elements is called a <u>transposition</u>.

It can be shown (and proved) that any permutation (in cyclic not.) can be decomposed into a <u>product</u> of transpositions.

Example:

(0 1 2) = (0 1) * (0 2)

$0 \rightarrow 1 \rightarrow 1$ (0 1

$1 \rightarrow 0 \rightarrow 2$ (0 1 2)

Any cycle of 3 elements can be represented as a product of 2 transpositions:

(abc) = (ab) * (ac)

Any cycle of 4 elements can be represented as a product of 3 transpositions:

(abcd) = (ab) * (ac) * (ad)

Thus, (0 1 2 3) = (01) * (02) * (03)

$0 \rightarrow 1 \rightarrow 1$ ~~(01)~~ (01
$1 \rightarrow 0 \rightarrow 2$ (012
$2 \rightarrow 0 \rightarrow 3$ (0123)

In general, a cycle of n elements has a representation as the product of n-1 transpositions.

[problem of identity as prime and "evenness"]

Parity, contd.

Note:

(1) the cyclic notation of a permutation is not unique

e.g., (012) = (120) = (201)

(2) The number of transpositions is not a fixed property of a permutation.

For example,

(02) = (01) * (02) * (12)

0 1 2 (02)

1 0 2 1 (11)

thus, (02) [and (11)]

(3) However, the parity of the number of transpositions for a given permutation is always the same.

If the number of transpositions is even [odd] the permutation is said to be even [odd].

The number of even permutations is the same as the number of odd permutations in a group.

If the permutation is even [odd] it contains an even [odd] number of order inversions.

[Display list of elements for permutation group of degree 4 with order inversions, etc. -- next page]

depends upon parity of n

Permutation multiplication - 3

Properties of a binary operation:

A binary operation ~~x~~ * is <u>commutative</u>* iff for

every $x, y \in S$, ~~x*x~~ $x * y = y * x$. [example: $3 \times 2 = 2 \times 3$]

A binary operation * is <u>associative</u> iff

for every $x, y, z \in S$, ~~xx*x~~ $(x * y) * z = x * (y * z)$

Note: grouping ~~xxxxxxx~~ essential since operation is binary

[Example: $(3 \times 2) \times 1 = 3 \times (2 \times 1)$]
[But: $(3 - 2) - 1 \neq 3 - (2 - 1)$

* NB. permutation group is non-Abelian

G

$*$

associative For any $x, y, z \in G$, $x * (y * z) = (x * y) * z$

identity : there is an identity element $i \in G$
with respect to $*$

For every $x \in G$, $x * i = i * x = x$

inverse for every $x \in G$ there is an inverse x' with respect
to $*$ such that

$x * x' = x' * x = i$

Groups

Definition of group:

A non-null set G with a binary operation * is a group with respect to * iff the following axioms are satisfied:

(1) * is associative in G

[For any $r, s, t \in G$, $r * (s * t) = (r * s) * t$]

~~[illegible]~~

(2) there exists an identity element i in G with respect to *

[For every $r \in G$, $r * i = i * r = r$]

(3) Every element g of G has an inverse g' with respect to *

[For any $r \in G$, there exists a unique element r' such that $r * r' = r' * r = i$]

Example of a group

A group can be conviently displayed in a group multiplication table such that the result of the binary operation can be easily read for each pair.

Group multiplication table for
the additive group of integers modulo 3

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Note that + is associative, there is an identity element (0), and every element has an inverse

A group is an instance of a mathematical model (or potential model). Advantages of models . . . .

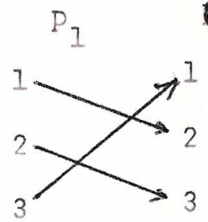
TABLE SHOWING INCREASE IN
NUMBER OF PARTITION-TYPES:

No. of Classes	#(S)	
2	2	2
3	3	2 + 1 3 itself
5	4	2 + 1 + 2 — 4 itself and one equal partition
7	5	2 + 1 + 2 + 2 - 5 itself and 1 new partition
11	6	2 + 1 + 2 + 2 + 4 6 itself, (1 equal 3-part partition, 1 equal 2-part partition) ~~2 equal partitions~~ and one new 2-part partition
15	7	2 + 1 + 2 + 2 + 4 + 4 7 itself, 2 new 2-part partitions and one new 3-part partition
22	8	2 + 1 + 2 + 2 + 4 + 4 + 7 8 itself, ~~2 equal partitions~~, ~~4 new partitions~~ 1 equal 4-part partition, 1 equal 2-part partition, 2 new 2-part partitions, 2 new 3-part partitions
30	9	2 + 1 + 2 + 2 + 4 + 4 + 7 + 8 9 itself, 1 equal 3-part partition, 1|4-part partition, 2|3-part partitions, 3|2-part partitions
42	10	2 + 1 + 2 + 2 + 4 + 4 + 7 + 8 + 12
56	11	
77	12	

Permutation multiplication -- the group operation in a permutation group

1-1 Notation of permutations (cyclic notation later)

$p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

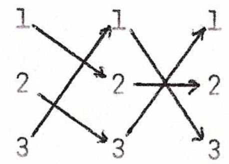


permutation displayed as a mapping (onto)

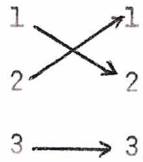
Consider two permutations

$p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

Consider the mapping p_1 followed by the mapping p_2



The result of the two mappings ~~may~~ can be represented as:



Or it ~~may~~ can be represented as the permutation $p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

SIGNIFICANCE OF INVERSE-RELATED PERMUTATIONS.

THE INVERSE OF A PERMUTATION "UNDOES" THE PERMUTATION WITH RESPECT TO THE PRIME, ORDERING:

> IF A PERMUTATION F_1 IS THOUGHT OF AS DERANGING A PRIME ORDERING, THEN THE INVERSE OF F_1 RESTORES THE PRIME.

IN THIS RECIPROCAL RELATION RESIDES THE INTUITIVE MEANING OF THE INVERSE RELATION FOR MUSIC. ELEMENTS OF THE SET ARE IN ONE-TO-ONE CORRESP. (REVERSIBLE) IN THE CASE OF INVERSE PAIRS.

But conjugate pairs more meaningful

WE CAN DISTINGUISH BETWEEN THOSE CASES IN WHICH A PERMUTATION IS ITS OWN INVERSE AND THOSE IN WHICH THE INVERSE IS DISTINCT.

> IN THE LATTER CASE THE CYCLIC NOTATION SHOWS THAT THE INVERSE-RELATED PAIR HOLD THE SAME ELEMENT(S) FIXED.

Class not yet covered

INVERSE-RELATED PERMUTATIONS BELONG TO THE SAME CLASS AND HAVE THE SAME NUMBER OF ORDER-INVERSIONS WITH RESPECT TO THE PRIME ORDERING AND WITH RESPECT TO EACH OTHER

THE INVERSE OF ANY PERMUTATION IS EASILY FOUND BY REVERSING THE MAPPING OF EACH ELEMENT

EXAMPLE:

~~0123~~
~~2310~~

~~0123~~
~~3201~~

a	a'
$0 \to 2$	$0 \to 3$
$1 \to 3$	$1 \to 2$
$2 \to 1$	$2 \to 0$
$3 \to 0$	$3 \to 1$

(cf. degenerate pc sets)

Partition of a Set

Definition: For a given set S, a collection of subsets of S such that the subsets are mutually disjoint (i.e., the intersection of every pair = $\emptyset$) and the union of the subsets = S (exhausts S) is called a partition of S.

Partitioning

Consider a set of two elements and a process of assigning those elements to disjoint subsets or cells. That is, we will not place any element in more than one cell. Let us begin by placing each element in a separate cell.

cells	a	b
	1	1
		2

We continue by placing 2 elements in one cell and distributing the others in unit cells and so on until all the elements are in one cell and the process is completed, as shown above for the 2-element case. This process, called partitioning, is shown below for 3, 4, and 5 element sets.

3 elements

a	b	c
1	1	1
	2	1
		3

4 elements

a	b	c	d
1	1	1	1
	2	1	1
		3	1
		2	2
			4

5 elements

a	b	c	d	e
1	1	1	1	1
	2	1	1	1
		3	1	1
		2	2	1
			4	1
			3	2
				5

These are unordered partitions.

Observe that these partitions do not show which elements are in the cells, the order of elements in the cells or the order of the cells.[This is important in the computation of the number of permutations for each partition (permutation class)]

Order relations - general remarks

Order relations can be investigated [most fruitfully] in terms of permutations, permutation groups, and partitions.

The examination of large-scale abstract structures provides insights into specific musical (compositional) instances, just as the study of relations between pitch-sets in terms of the total collection of such sets provides insights into the ways in which these sets occur in individual works.

---

use Babbitt's ordered pair (a, b)

where a = pc

b = order-position

Introductory

COMPARISON OF UNORDERED AND ORDERED PITCH-SETS IN THE 12-PITCH SYSTEM

SEE revised computation for 220 pc sets

CARDINAL NUMBER OF SET	NUMBER OF DISTINCT UNORDERED SETS		NUMBER OF DISTINCT ORDERINGS (PERMUTATIONS)
12	1	1 · 12!=	479,001600
11	12	12 · 11! =	479,001,600
10	66	66 · 10! =	239,500,800
9	220	220 · 9! =	79,833,600
8	495	495 · 8! =	19,958,400
7	792	792 · 7! =	3,991,680
6	924	924 · 6! =	665,280
5	792	792 · 5! =	95,040
4	495	495 · 4! =	11,880
3	220	220 · 3! =	1320
2	66	66 · 2! =	132
1	12	12 · 1! =	12
0	1	1 · 0! =	1

TOTAL 1,302,060,157

THUS, ESSENTIAL TO REDUCE NUMBER OF PERMUTATIONS BY CLASSIFYING, SHOWING DISTINCTIVE AND NON-DISTINCTIVE CHARACTERISTICS AT A HIGHER LEVEL OF STRUCTURE.

```
1 * 12 FACTORIAL= 479001600
1 * 11 FACTORIAL= 39,916,800
6 * 10 FACTORIAL= 21,772,800
12 * 9 FACTORIAL= 4,354,560
29 * 8 FACTORIAL= 1,169,280
38 * 7 FACTORIAL= 191,520
50 * 6 FACTORIAL= 36,000
38 * 5 FACTORIAL= 4,560
29 * 4 FACTORIAL= 696
12 * 3 FACTORIAL= 72
6 * 2 FACTORIAL= 12
12 * 1 FACTORIAL= 12
1 * 0 FACTORIAL= 1
```

Sum = 546,447,913

Comparison of Unordered and Ordered Pitch-Class Sets

The following table indicates the desirability of reducing the number of permutations (ordered sets) in some way.

Cardinal Number of Set	Number of Distinct Unordered Sets	Number of Ordered Sets (Permutations)
12	1	1 x 12! = 479,001,600
11	1	39,916,800
10	6	21,772,800
9	12	4,354,560
8	29	1,169,280
7	38	191,520
6	50	36,000
5	38	4,560
4	29	696
3	12	72
2	6	12
should be 1 — 1	12	12 — 1
0	1	1
	Total =	546,447,913

The number of permutations in a given permutation-class (partition-type).

This is analogous to the number of pitch-class sets derived from the prime forms by inversion and transposition.

Let $N(P_i)$ represent the number of permutations in a permutation class P_i.

For example, $P_i = 111$

The number $N(P_i)$ is computed according to the following expression:

$$N(P_i) = \prod_{j=1}^{k} \left(\frac{C_j!}{C_j} \right) \left(\frac{S(M, C_j)}{E} \right)$$

or $n(p_i) = \prod_{j=1}^{k} (c_j - 1)! \cdot (s(m, c_j)/E)$

$\prod$ is ordinary product

k is the number of cells in the partition.

$\frac{C_j!}{C_j} = (C_j - 1)!$ is the number of orderings of the cell C_j, excluding circular orderings, which are necessarily equivalent.

$S(M, C_j)$ is the number of unordered subsets (combinations) of cardinal number C_j in a set of M elements, where $M = C_j + C_{j+1} + C_{j+2} + \ldots + C_{j+(k-1)}$ [i.e., the sum of the cells from C_j through the remaining cells.] ⊗

E is the number of cells of cardinality C_j among the cells $C_j, C_{j+1}, C_{j+2}, \ldots, C_{j+(k-1)}$

⊗ $C(n,m) = \frac{n!}{m!(n-m)!}$

Number of permutations in each class continued:

Sample computation:

$P_i = 221$

$$\underbrace{2!/2 \times 10/2}_{j=1} \times \underbrace{2!/2 \times 3/1}_{j=2} \times \underbrace{1!/1 \times 1/1}_{j=3} = 15$$

Another one:

$P_i = 22111$

$$\underbrace{2!/2 \times 21/2}_{j=1} \times \underbrace{2!/2 \times 10/1}_{j=2} \times \underbrace{1!/1 \times 3/3}_{j=3} \times \underbrace{1!/1 \times 2!/2}_{j=4} \times \underbrace{1!/1 \times 1/1}_{j=5} = 105$$

Interpretation of Order Relations in terms of interval succession

By interval succession we mean simply the intervals formed by successive pitches. For example,

0 1 3 yields the interval succession 1 2

It is interesting to consider the relation between the total interval-content of a set and the number of intervals that can be stated "directly" -- i.e., as linear intervals.

Cardinal Number	Intervals	No. of Linear Intervals
2	1	1
3	3	2
4	6	3
5	10	4
6	15	5
7	21	6
8	28	7
9	36	8
10	45	9
11	55	10
12	66	11

Obviously, for a set of cardinality n, only n-1 intervals can be stated directly.

[Function of repetition in compositional sets may be way of bringing in other intervallic relations]

Linear Sets: Special Properties

ORDER-RELATIONS

Number of notes n in set	Number of Successive Intervals =(n - 1)	Total Number of Intervals	Number of Intervals not directly stated
3	2	3	1
4	3	6	3
5	4	10	6
6	5	15	10
7	6	21	15
8	7	28	21
9	8	36	28
10	9	45	36

Comments:

1. Any horizontally stated (pitch-) set represents a partitioning (induces a partitioning) of the total intervallic content into a set of directly stated (successive) intervals and a set of indirectly (stated) (non-consecutive) intervals.

2. The ratio of directly stated to indirectly stated intervals varies inversely with size of set.

3. The vectors of the directly stated set and the indirectly stated set may or may not be members of the set of vectors of total intervallic content for the particular set-size.

4. (Musical) Significance may lie in answer to question: Which intervals are directly stated as compared to those indirectly stated?

5. A ~~linear~~ set of n elements has n! possible orderings. But, the number of different interval-vectors for a particular set is considerably smaller. [See illustration of different vectors for 4-2, 3-3, and 4-28.]

6. In view of #5, it appears that"unordered"linear sets might be compared conveniently on the basis of their interval vectors.

7. To do: Make list of linear vectors for each set (~~COMPUTER!!!~~) and discover mathematical basis.

done: basic interval patterns (to be completed for hexachords)

Interpretation of Order-inversions with respect to permutation classes

[List (computed) showing correspondence of permutations and permutation-classes]

[Table showing number of order inversions and corresponding number of permutations for degrees ~~x~~ 3, 4, 5] — separate page

General observations on former:

(1) In every case the greatest number of permutations corresponds to the class (n-1)(1). Implications for historical development of a focal note (?)

but check actual permutations in each class for, e.g., degree 4

SEE material on permutation classes for relation of permutations and partitions

SEE also notes on conjugate classes

Explanation of numerical basis of partitioning:

Let us ~~assume~~ consider a set of two elements

[illegible] and a partitioning (or assigning) of these elements into disjunct subsets or cells.

This set may be partitioned beginning with the assignment of each element to ~~a~~ an individual cell (a partition which is determined, since the number of availalbe elements corresponde to the number of available cells. From thence we can proceed to combine the elements systematically, gradually reducing the number of unit cells (cells containing only one element), until the partitioning reaches the other extreme: viz., all the elements are in one cell. This procedure may be diagrammed for sets of various numbers of elements, as follows:

These are, more correctly, "unordered partition-types": they do not show which elements are in the cells, the order of elements in the cells, or the order of the cells.

2 elements

cells	a	b
elem.	1	1
		2

3 elements

cells	a	b	c
elem.	1	1	1
		2	1
			3

4 elemtns

cells	a	b	c	d
elem.	1	1	1	1
		2	1	1
			3	1
			2	2
				4

5 elem.

cells	a	b	c	d	e
elem.	1	1	1	1	1
		2	1	1	1
			3	1	1
			2	2	1
				4	1
				3	2
					5

6 elem.

cells	a	b	c	d	e	f
elem.	1	1	1	1	1	1
		2	1	1	1	1
			3	1	1	1
			2	2	1	1
				4	1	1
				3	2	1
				2	2	2
					5	1
					4	2
					3	3
						6

7 elements

cells	a	b	c	d	e	f	g
elem.	1	1	1	1	1	1	1
		2	1	1	1	1	1
			3	1	1	1	1
			2	2	1	1	1
				4	1	1	1
				3	2	1	1
				2	2	2	1
					5	1	1
					4	2	1
					3	3	1
					3	2	2
						6	1
						5	2
						4	3
							7

Let $N(p_i)$ designate the number of partition-types for a set containing i elements.

SEE diagram [illegible] diagram (for general equation)

Then, from the above,

pairs of even and odd numbers of elements

$N(p_2)=N(p_1)+1=2$

$N(p_3)=N(p_2)+1=3$

$N(p_4)=N(p_3)+2=5$

$N(p_5)=N(p_4)+2=7$

number of partitions composed of odd [illegible]

$N(p_6)=N(p_5)+4=11$

$N(p_7)=N(p_6)+4=15$

$N(p_8)=N(p_7)$ ~~+ 2 = 21~~ +7 = 22

$N(p_9)=N(p_8)+$ ~~[illegible] 2~~ +8 = 30

$N(p_{10})=N(p_9)+$ ~~[illegible]~~ +12 = 42

$N(p_{11})=N(p_{10})$ +14 = 56

$N(p_{12})=N(p_{11})$ +21 = 77

(Diagram of partitions of 12 elements on ~~following pages~~ ~~another page~~)

(also basis of distinct sets!)