TABLE 2

GRP-TBL.3

Permutation Group of Degree 3

∘	π_1	π_2	π_3	π_4	π_5	π_6
π_1	π_1	π_2	π_3	π_4	π_5	π_6
π_2	π_2	π_3	π_1	π_5	π_6	π_4
π_3	π_3	π_1	π_2	π_6	π_4	π_5
π_4	π_4	π_6	π_5	π_1	π_3	π_2
π_5	π_5	π_4	π_6	π_2	π_1	π_3
π_6	π_6	π_5	π_4	π_3	π_2	π_1

Linear Features of Debussy's *Nuages* (Trois Nocturnes)

Introduction (to follow)

In this chapter new features of linearity are introduced in the context of a large-scale composition by Debussy--e.g., permutation multiplication. Possibly include group multiplication table for degree 4 (check out first).

[Table and example numbers will have to be changed, perhaps using a descriptor for the chapter.]

[This material presumes the Schoenberg Op. 15 comes first. If not, the heuristics developed there, as well as the material on order relations and the duplicate reduction algorithm will have to be given separately elsewhere or repeated here for Debussy--to the extent that they are used, of course.]

Permutations of Degree Three

As was the case in Chapter x order relations deserve analytical attention in *Nuages* and, indeed, in Debussy's music altogether. Because the thematic motives of *Nuages* are often trichords, as I will demonstrate, it is essential to have at hand a list of permutations of degree three. Table 1 provides the basic information.

Table 1

On the basis of partition-types the six permutations of degree three group into three classes, labelled A, B, and C on Table 1.

Class A has only one member, in which each cell is mapped onto itself. This may also be described as a circular permutation from cell 0.

Class B permutations, of which there are two, are both circular, thus preserving ordered adjacency, even though all three cells change positions. Compared to Class B permutations, Class C permutations represent a markedly greater degree of derangement. PRM4 holds cell 2 fixed while reversing the positions of cells 0 and 1. PRM5 (retrograde) reverses the order of cells 0 and 2 while holding cell 1 fixed. PRM6 holds cell 2 fixed while exchanging cells 0 and 1.

Permutation Multiplication

Table 2

Table 2 is a "group multiplication" table. It shows the permutation that results when one permutation is applied, then another. The asterisk in the upper left corner of the table represents this compound operation, which is called permutation multiplication. As before, the letters PRM stand for permutation, and the number after each PRM corresponds to the number of the permutation on Table 1. Let us consider an example of permutation multiplication, beginning, arbitrarily, with PRM2(3-7), as it appears on Table 1:

PRM2 (012) C# F# E
Now we apply PRM4 (01) F# C# E
Followed by PRM3 (021) C# E F#

Here the final result (called the product) is the same as PRM5 applied to PRM2, and on Table 2 PRM5 occurs at the intersection of the row that begins on the left with PRM4 and the column that begins at the top with PRM3, which is to say that PRM4 followed by PRM3 produces PRM5.

Axioms of Group Theory [N1: after R.B. Kershner and L.R. Wilcox, *The Anatomy of Mathematics*. New York: The Ronald Press Company, 1950, 83. The following assumes some familiarity with elementary mathematics and mathematical terms. It is not absolutely essential, however, to understand this formal material in order to follow the later discussions of the relevant linear order relations.]

The permutation group displayed in Table 2 satisfies the axioms of group theory listed below, in which G is a set and o an operation on G X G to G. (Lower case e replaces the conventional epsilon in the expressions that follow.)

These axioms are to be checked again.

I. For every a,b,c e G (a o b) o c = a o (b o c).
[associativity]

II. For every a,b e G, there exists x e G such that
a o x = b.

III. For every a,b e G, there exists y e G such that
y o a = b. [closure]

Observe that each row and column of the group table proper (Example 2), excluding row 1 and column 1, the constituents of which serve as identifiers, contains the complete array of permutations, with each represented only once. This satisfies Axioms II and III. When multiplication is extended to three permutations, the same product is obtained if the multiplication is first performed on the first two and then the third, or on the last two and then the first, as indicated by the parentheses in Axiom I above, satisfying the axiom of associativity.

In addition, the identity element of G is defined as follows:

Let i denote the unique element of G such that,

for every a e G, a o i = i o a = i.

The Identity element and inverse-related permutations

From inspection of Table 2 it can be determined that PRM1 is the unique identity element: The result of any multiplication performed on operands PRMx and PRM1 is PRMx, where x = 1, 2, . . . , 6.

From axioms II and III it follows that the identity permutation PRM1 must be the product of every pair of elements a, b in the group, including the situation in which a = b. These correspondences, or inverse-related pairs present a relatively simple picture in the permutation group of degree three: each permutation is its own inverse, with the exception of PRM2 and PRM3, which are inverse related. Thus, for example, the application of PRM5 followed by the application of PRM5 will produce the identity permutation PRM1. On the table this is represented by the appearance of PRM1 at the intersection of row 5 and column 5.

Finally, note that the order in which the operands are multiplied is significant. Thus, PRM4 followed by PRM3 produces PRM5, while PRM3 followed by PRM4 produces PRM6. (The operation permutation multiplication is not commutative.)

As we proceed, we will frequently consider order relations and extend to a consideration of the multiplication operation, including inverse-related pairs, which "undo" each other.

In general, what the group model of permutations offers is a closed universe, a network of interrelations. [Quote Lewin?]
Footnote on Berg Lyric Suite movement with perms. From the standpoint of the group structure, designation of PRM1 is essentially arbitrary. From the analytical standpoint I have tried to be consistent in selecting as PRM1 the first temporal occurrence of a motive (as in Ex. 2a below).

Include subgroups?

Thematic Motives in *Nuages*

Following the general thrust of this study so far, which assumes that motivic and linear structures are intimately connected, Example 1 presents the first theme (Theme 1a) of *Nuages*.

Example 1

Assuming that the horizontal dimension is primary here, the theme appears to be essentially a bi-linear configuration, which of course does not imply that vertical coincidences are unimportant. [N2 Parks as well] Thus, the totality of the upper line is a form of what some Hungarian writers on Bartók's music call the "diatonia seconda," heptad 7-34, which combines octatonic and whole-tone features, while the lower line (omitting chromatic passing tones) is octatonic 7-31 from Collection I. These two heptads unfold contrapuntally, following a measure-by-measure pattern.

A prominent feature of the theme to which I shall draw attention is its metrical organization. The 6/4 meter signature suggests a duple grouping: 3 + 3 quarter notes. The regular contour of descending dyads, however, suggests a triple grouping: 2 + 2 + 2. In this situation, I propose a mediating explanation: the duple is harmonic in nature, applying to the relation between the two lines, while the triple is linear-melodic, articulating the leapfrog parsing to be presented below.

Duplicate Reduction: tetrachordal and trichordal conjunction

The extraction of linear motives from the upper line of the bi-linear theme obviously offers a number of possiblities, enhanced by the equal durational values for the first pattern (bar 1, with repetition in bar 2) and the "simple" four quarters-half pattern of bars 3 and 4. Let us therefore proceed systematically, beginning with the duplicate reduction algorithm, which, in effect, ignores the issue of metric-rhythmic organization. Example 2a gives the tetrachordal segmentation generated by the duplicate reduction algorithm.

Example 2a

The first segment produced by the algorithm is a form of diatonic tetrachord 4-11 that spans the entire pattern, rejecting duplicates of the first two notes. Since this is the first conjunct form of 4-11 (without duplicates), I have designated it both PRM1 and T_0. The two permutations of 4-11 that follow, PRM8 and PRM9, in that chronological order, are both of Class C, permutations that hold two elements fixed. The immutable shared note in all three forms is of course the last note in m. 2, $c^{\#2}$, which is a special pitch and pitch-class representative in *Nuages*.

Example 2b

Duplicate reduction for trichords yields quite simple, yet interesting, results: three forms of trichord 3-6 and one of 3-2. The Class C permutations of 3-6, PRM6 and PRM4, constitute a considerable rearrangement of the basic form of 3-6 here, while, with respect to their internal relation, they are close. Table 2 shows that PRM6 (Row 6) followed by PRM4 (Column 4) results in PRM3 one of the three circular permutations that preserve adjacency of two elements (Table 1), here $f^{\#2}$ and d^2.

The circumstance that mm. 3-4 of the upper line of theme 1a (Example 1) contain no duplicates whatsoever--and, indeed, these "consequent" measures differ markedly from mm. 1-2 in other important respects--suggests that other heuristics may be applied to extract significant motivic components. Thus, Example 3 shows the results of the interlocking heuristic applied to tetrachords, trichords, and pentads.

Example 3

Ex. 3a: Parsing by interlocking tetrachords reveals that motives based upon only two set classes are produced: 4-z15 and 4-10. Here as elsewhere the analytical portion of this study will explore the ways in which these thematic fragments relate to linear and other formations in the work.

Ex. 3b: With respect to set class the interlocking trichords of the second part of the first theme (mm. 3-4) are similarly restricted, yielding only sets of classes 3-3 and 3-7, and precisely two representatives of each of them. Although both trichords participate in the motivic organization of *Nuages*, trichord 3-3 is perhaps more prominent in this portion of the first theme of the work because it comprises the disjoint three-note groups that culminate on the last note of the theme, g^1, at m. 5. The second of these trichords, beginning on $g^{\#1}$, is as retrograde inversion of the first; to be precise, it is $PRM_5(T_5I)$. As perhaps the most apparent consequence of this relation the interval succession of the second form of 3-3 reverses that of the second: -4 +3 becomes +3 -4.

Ex. 3c: With the pentadal parsing the reason for the limited number of motivic set classes in Examples 3a and 3b becomes evident, for here the underlying octatonic structure of the second part of the theme becomes evident. This consists of a reordering of an ordered octatonic scale segment. [Note on octa article in Music Analysis and D.'s use of both ordered and "unordered" octatonic collections.] The cyclic notation of 5-10 in m. 3, with respect to the ordered form of 5-10 is (0324)(1)--i.e., only a# retains its position in the referential ordered collection. In m. 4, because of the extension of the line to the final g^1 in m. 6, a second form of octatonic 5-10 is created. Its cyclic notation with respect to the referential ordering is (0412)(3), a member of the same permutation class as the first form, namely, 41, of which there are 30 members (the largest class of permutations of degree 5).

Reflecting the underlying symmetric octataonic harmony, surface symmetries abound in the final phrase of the first theme (m. 4 through the first note of m. 5). The disjunct trichords discussed in connection with Example 3b, which sum to an ordered use of both ordered and "unordered" octatonic collections.] The cyclic notation of 5-10 in m. 3, with respect to the ordered form of 5-10 is (0324)(1)--i.e., only a# retains its position in the referential ordered collection. In m. 4, because of the extension of the line to the final g^1 in m. 6, a second form of octatonic 5-10 is created. Its cyclic notation with respect to the referential ordering is (0412)(3), a member of the same permutation class as the first form, namely, 41, of which there are 30 members (the largest class of permutations of degree 5).

Reflecting the underlying symmetric octataonic harmony, surface symmetries abound in the final phrase of the first theme (m. 4 through the first note of m. 5). The disjunct trichords discussed in connection with Example 3b, which sum to an ordered linear form of octatonic hexachord 6-z13, present the retrograde intervallic symmetry -4 +3 +3 -4, as noted above. When the middle interval of this phrase, -5, is included it becomes the axis of a symmetry that spans the entire configuration, as shown in mm. 4-5 of Example 3b. The extent to which these symmetric features contribute to the motivic organization of other parts of the composition will be considered at a later time.

Finally, the music-pictorial opposition of diatonic and octatonic harmonies in the first theme, with its two contrasting parts is inescapable. In *Nuages*, as in many other of his works, the composer achieves . . . .

Application of the Leapfrog Heuristic

The consistent contour pattern of descending dyads separates the components of the first part of the upper line of Theme 1 into two strands. Regarding this as an analytical imperative, we apply the leapfrog heuristic, modified by appending to each strand the tailnote of the melodic phrase in order to complete a trichord.

Example 4

Example 4 shows the results of this parsing. Perhaps the most striking feature it reveals is the series of 3-2 trichords in the lower linear component. This class of trichord is not new to our analysis; it occurred twice in the duplicate reduction parsing (Example 2b). Here in Example 4, however, it spans the upper component of the entire theme, summing to octatonic 7-31 of Collection I, as indicated. When the leapfrog heuristic is applied to the upper component of the theme it produces a purely diatonic pentad, 5-23, familiar as the lower pentad of the minor scale. Thus, in the leapfrog segmentation, we again perceive an opposition of distinctly different harmonic entities *within the theme*: diatonic and octatonic, in which the 3-2 trichords play a mediating role, combining in the final two measures to form octatonic hexachord 6-z13.

The lower line of Theme 1/1a?

The three examples grouped under Example 5 display the parsing of the lower line of Theme 1.

Example 5a

Example 5a illustrates the results of a compound parsing: chromatic reduction followed by duplicate reduction. Although two chromatic reductions are possible--omitting either
b^{b1} or a^1--I have shown only the latter, regarding a^1 as a "neighbor" connecting b^{b1} with its enharmonic equivalent $a^{\#1}$. This choice produces trichord 3-2, which has already tentatively achieved motivic status, whereas reducing out b^{b1} produces 4-2,

which would prove to be a dubious motivic component in *Nuages*, I believe. [I realize that the reasoning here is somewhat circular, but I couldn't help it.]

Duplicate reduction applied to the chromatic reduction yields three forms of motivic 3-2, the second and third of which are PRM6-related to the first. This permutation, it will be recalled, reverses the first two cells while holding the third fixed in position. Another way of looking at this compound parsing is to observe that the three forms of 3-2 produced by the duplicate reduction heuristic are subsegments of the 5-note segment produced by chromatic reduction alone. Thus, this portion of the lower counterpoint (parsed in this way) is saturated by instances of set class 3-2, which, as remarked above, has both octatonic and diatonic affiliations.

Parsing of the first part of the lower line of Theme 1 by interlocking and leapfrog heuristics does not produce interesting results. As the reader can easily ascertain, interlocking yields two forms of 3-2 that are ordered pitch-class equivalents of the sets already extracted by the duplicate reduction algorithm, while the leapfrog heuristic (with shared end note) dissects the line into two dyads, b^{1}-$a^{\#1}$ and $c^{\#2}$-$a^{\#1}$, which are unlikely candidates for motivic status.

Example 5b

Although m. 2 of the lower strand of the first theme of *Nuages* varies from m. 1 of the same strand by only one note, the last, the result of the chromatic/duplicate parsing is strikingly different, as shown in Example 5b. Specifically, the chromatic reduction and the first two stages of the duplicate reduction algorithm yield the same pitch form of tetrachord 4-10, a formation that has both octatonic and diatonic connections. Of these, the one produced by the second stage of the duplicate reduction (RSEG5) is the most potent, since it contains two inversionally related forms of trichord 3-2, which is a strong candidate for the primary three-note motive in the work. [MORE COMMENTS ON PITCH-ORDER RELATIONS IF THESE ARE GERMANE TO LINEAR-MOTIVIC STRUCTURES LATER ON.]

Example 5c

And although, again, the application of the leapfrog heuristic does not yield strikingly new information concerning the internal organization of this portion of the lower strand of Theme 1, producing only two disjunct dyads, the interlocking heuristic applied to the first form of 4-10 derived by means of the duplicate reduction algorithm does reveal a new trichord: 3-7, as shown in Example 5c.

Example 5d

Application of the compound transformation Chromatic reduction followed by the duplicate reduction algorithm produces the surprising result shown in Example 5d.

Include list of permutation sequence

Selection of f# rather than f to omit is determined by the operative octatonic collection here: Coll. I (check)

Harmonic Motives (motives formed by grouping linear segments)
and, especially, cadential b over g, which plays an axial role in the music.

See annotated old Example 1.

Horizontal and Vertical Dyads

Discuss other parsings: especially contour (pre-empted by previous heuristics--e.g., contour by duplicate reduction)

CHECK OLDER EXAMPLES OF THEME 1 NOW AND DISCUSS OTHER PARSING HEURISTICS BEFORE PROCEEDING TO THEME 2

Theme 2

Boundaries of Themes 1 and 2, etc.

INVENTORY OF MOTIVES--WITH SHORT EXAMPLES OF THEIR OCCURRENCES IN THE LINEAR DIMENSION (ALREADY DONE)

LINEAR-MOTIVIC ANALYSIS OF ENTIRE WORK, SECTION BY SECTION.